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Exact solutions of Fisher and Burgers equations with finite transport memory

Sandip Kar, Suman Kumar Banik and Deb Shankar Ray

Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700 032, India

E-mail: pcdsr@mahendra.iacs.res.in

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Abstract

The Fisher and Burgers equations with finite memory transport, describing reaction–diffusion and convection–diffusion processes, respectively have recently attracted a lot of attention in the context of chemical kinetics, mathematical biology and turbulence. We show here that they admit exact solutions. While the speed of the travelling wavefront is dependent on the relaxation time in the Fisher equation, memory effects significantly smoothen out the shock wave nature of the Burgers solution, without any influence on the corresponding wave speed. We numerically analyse the ansatz for the exact solution and show that for the reaction–diffusion system the strength of the reaction term must be moderate enough not to exceed a critical limit to allow a travelling wave solution to exist for appreciable finite memory effect.

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1. Introduction

A number of nonlinear phenomena in physical [1], chemical [2] and biological processes [3, 4] are described by the interplay of reaction and diffusion or by the interaction between convection and diffusion. The well-known partial differential equations which govern a wide variety of them are the Fisher [5] and Burgers [6] equations, respectively. While the Fisher equation describes the dynamics of a field variable subject to spatial diffusion and logistic growth, the Burgers equation provides the simplest nonlinear model for turbulence. Since spatial diffusion is common to all these processes, Fick's law forms the key element in the description of transport. This description, however, gets significantly modified when the memory effects are taken into account, i.e. when the dispersal of the particles is not mutually independent. This implies that the correlation between the successive movements of the diffusing particles may be understood as a delay in the flux for a given concentration gradient. Over the last few years the analysis of memory effects in diffusive processes has attracted a lot of attention

[7–23] in chemical kinetics, mathematical biology and allied areas. The focal theme lies in the interesting travelling wavefront solutions and related issues which have been studied extensively by several authors in various recent contexts, which include particularly the effect of noise and stochasticity in the microscopic stochastic model [26], interface equations [27], random fragmentation problem [28], nonuniform reaction rate distribution [29], autocatalytic front [30], epidemic model [31] etc. The objective of the present paper is to show that the Fisher equation and the Burgers equation with finite memory transport admit exact solutions. We numerically clarify the nature of the ansatz wherever necessary and analyse the physical implications of the solutions modified by relaxation effects and the related issues.

2. The Fisher and Burgers equations with finite memory transport

The starting point of our analysis is Cattaneo's modification [24] of Fick's law in the form:

$$J(x, t+\tau) = -D\frac{\partial u(x,t)}{\partial x} \tag{1}$$

which takes care of adjustment of a concentration gradient at time t with a flux $J(x, t+\tau)$ at a later time $(t+\tau)$ and τ being the delay time of the particles in adopting one definite direction of propagation. Here $u(x, t)$ denotes the field variable, and D is the diffusion coefficient of the particles.

The population balance equation for the particles, on the other hand, takes into account conservation of the equation supplemented by a source function $kf(u)$ for the particles in the form

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J}{\partial x} + kf(u). \tag{2}$$

The Fisher source function $f(u) = u(1-u/K)$ has been the subject of wide interest in various contexts. Here the first term in $f(u)$ signifies linear growth followed by a nonlinear decay due to the second one; k and K being the growth rate constant of the population and the carrying capacity of the environment, respectively. In what follows we shall consider two specific cases of the flux-gradient relation (1) for the Fisher and Burgers problem.

2.1. The Fisher equation with nonlinear damping and finite transport memory

We start with an expansion of J in equation (1) [25] up to first order in τ to obtain

$$\tau\frac{\partial J(x,t)}{\partial t} + J(x,t) = -D\frac{\partial u}{\partial x}. \tag{3}$$

Here $u(x, t)$ represents the density function. Differentiating (3) with respect to x and differentiating (2) with respect to t and eliminating J from the resulting equations, one has

$$\frac{\partial^2 u}{\partial t^2} + [\beta - kf'(u)]\frac{\partial u}{\partial t} = \beta kf(u) + w^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

where we have used the following abbreviations

$$\beta = 1/\tau \qquad \text{and} \qquad w^2 = \beta D. \tag{5}$$

Equation (4), a hyperbolic reaction–diffusion equation, is a generalization of the Fisher equation for finite memory transport and nonlinear damping. It reduces to the standard Fisher equation for $\tau = 0$. Over the years the equation has drawn wide interest in the context of travelling wave solutions in various problems [7–23]. For example, Gallay and Raugel [8, 9] have studied the propagation of a front solution without the nonlinear term $kf'(u)$.

Horsthemke has discussed some related issues in the problem of transport-driven instabilities [16].

We now look for the travelling wave solutions of equation (4) of the form $u(x,t) = KU(x-ct) \equiv KU(z)$ with $z = x - ct$, where $c > 0$ is the speed of the nonlinear wave (which, in general, is different from the linear wave w dictated by the medium subject to the boundary conditions:

$$U(-\infty) = 1 \qquad \text{and} \qquad U(+\infty) = 0. \tag{6}$$

Equation (4), therefore, after some algebra assumes the following form

$$\frac{\partial^2 U}{\partial z^2} + [c(n-A)]\frac{\partial U}{\partial z} - 2AcU\frac{\partial U}{\partial z} + nAmU(1-U) = 0 = L(U) \quad \text{(say)} \tag{7}$$

where

$$m = w^2 - c^2 \qquad \text{and} \qquad n = \beta/m \qquad \text{and} \qquad A = k/m. \tag{8}$$

Following Murray [4] we now introduce the ansatz,

$$U(z) = \frac{1}{[1 + a\exp(bz)]^s} \tag{9}$$

as a solution to equation (7), where a, b and s are positive constants to be determined. Using (9) in (7) we obtain after some algebra

$$\begin{aligned}
&[s(s+1)a^2b^2 + nAma^2 - s[c(n-A)]a^2b - sa^2b^2]\,\mathrm{e}^{2bz} \\
&\quad + [2aAmn - sab^2 - s[c(n-A)]ab]\,\mathrm{e}^{bz} + nAm \\
&\quad - 2Acsab\,\mathrm{e}^{bz}(1 + a\,\mathrm{e}^{bz})^{-s+1} - nAm(1 + a\,\mathrm{e}^{bz})^{-s+2} = 0 = L(U).
\end{aligned} \tag{10}$$

Now for $L(U) = 0$ for all z, the coefficients of e^0, e^{bz}, e^{2bz} and e^{3bz} within the appropriate brackets must vanish identically. This implies that $s = 0$, 1 or 2. $s = 0$ is not a possible solution since s is a positive constant by our starting assumption. For $s = 1$, the coefficients of e^{bz} and e^{2bz} of equation (10) yield the following relations,

$$s(s+1)b^2 + nAm - s[c(n-A)]b - sb^2 = 0 \tag{11}$$

$$nAm - sb^2 - s[c(n-A)]b - 2Acsb = 0 \tag{12}$$

which can be solved to give $b = 0$ and $b = -2Acs/(s+1)$.

Again, since by initial assumption b is a positive constant, both the values of b are unacceptable and $s = 1$ is not a correct choice.

For $s = 2$, equation (10) reduces to a form in which the coefficients of e^{bz}, e^{2bz} and e^{3bz} must satisfy the following relations

$$s(s+1)b^2 + 3nAm - 2s[c(n-A)]b - 2sb^2 = 0 \tag{13}$$

$$s(s+1)b^2 + nAm - s[c(n-A)]b - sb^2 = 0 \tag{14}$$

and

$$2nAm - sb^2 - s[c(n-A)]b - 2Acsb = 0. \tag{15}$$

From equation (13)–(15) we obtain

$$b^2 = \frac{nAm}{s(s+1)} \tag{16}$$

and putting $n = \beta/m$, $A = k/m$ from (8) and $s = 2$ in (16) we have

$$b^2 = \frac{\beta k}{6m}. \tag{17}$$

Making use of (17) in (14) we obtain b in terms of c as follows:

$$c = \frac{5k\beta}{6b(\beta - k)} \tag{18}$$

$$b = \frac{5}{6c\left(\frac{1}{k} - \frac{1}{\beta}\right)}. \tag{19}$$

The exact speed c of the travelling wave can be calculated from (17) and (19) using $m = w^2 - c^2$ as

$$c = \frac{\sqrt{\beta D}}{\left[1 + \frac{6}{25}(y - 1/y)^2\right]^{1/2}} \tag{20}$$

with $y = \sqrt{\frac{\beta}{k}}$. It may be noted that the exact value of c thus derived is always greater than $c_{\min}$ where

$$c_{\min} = \frac{w}{\left[1 + \frac{1}{4}(y - 1/y)^2\right]^{1/2}}. \tag{21}$$

Again in the diffusive limit, i.e. $1/\beta \to 0$ or $1/y \to 0$, the expression (20) results in the exact Fisher value of c as $c = 5\sqrt{kD}/\sqrt{6}$. We note that this value of c is not too far from the $c_{\min}$ which is given by $c_{\min} = 2\sqrt{kD}$ as pointed out earlier by Murray [4].

Having determined b and s one can write down the exact form of the travelling wave solution (9) for the problem

$$U(z) = 1 \Bigg/ \left\{1 + a \exp\left[\left(\frac{5}{c\sqrt{6}\left(\frac{1}{k} - \frac{1}{\beta}\right)}\right)\frac{z}{\sqrt{6}}\right]\right\}^2. \tag{22}$$

Furthermore, a can be determined from the usual condition $U(z) = 1/2$ for $z = 0$. This results in $a = (\sqrt{2} - 1)$. The exact solution of the Fisher equation can be recovered from (22) in the limit $1/\beta \to 0$ (i.e. $1/y \to 0$) using the Fisher value of $c = 5\sqrt{kD}/\sqrt{6}$. This is given by

$$U(z) = \frac{1}{\left[1 + (\sqrt{2} - 1)\exp\left(\sqrt{\frac{k}{D}}\frac{z}{\sqrt{6}}\right)\right]^2}. \tag{23}$$

We thus observe that the effect of memory or finite relaxation time enters the dynamics of the reaction–diffusion system through its influence on the speed of the travelling wavefront c. We emphasize here that for $\frac{1}{\beta} = 0$ equation (22) does not give the solution selected by the front but is much steeper although the speed is very close to the selected one.

It is pertinent to point out that although exact, the travelling wave solution (22) does not exhaust the possibility of other solutions. This was noted earlier by Murray [4] in the context of the Fisher equation without memory effect which is a parabolic differential equation. For an understanding of the nature of the travelling wave solution where $\beta = (1/\tau)$ is a new element of the present theory, we carry out a numerical investigation of equation (4) using the finite difference method to solve the boundary value problem. The initial condition to integrate numerically is that the front is at rest at $t = 0$. We fix the value of the diffusion coefficient $D = 1.0$ for the entire treatment. In order to allow the variation of τ for a fixed value of k, we have kept k at 0.6. For a higher value of k, i.e. where the reaction term dominates, τ must be chosen

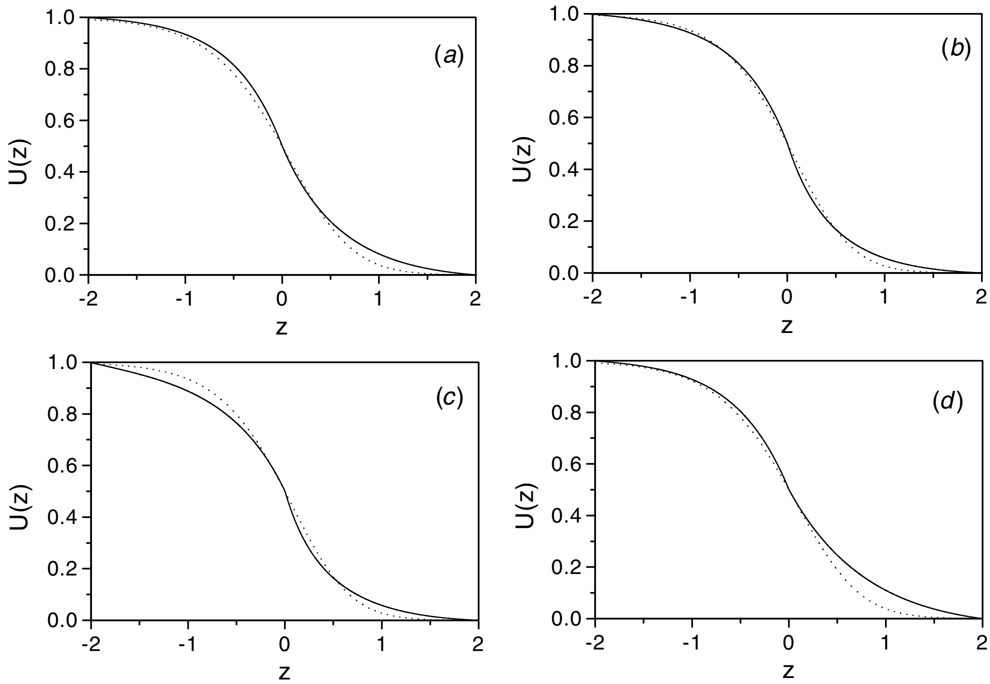

Figure 1. A plot of travelling wave solutions for different values of relaxation time $\tau(=\frac{1}{\beta})$ for $k = 0.6$ and $D = 1.0$. The solid lines are due to numerical simulations of equation (4) and the dotted lines are the analytic results (22). (*a*) $\tau = 0.2$, (*b*) $\tau = 0.4$ (*c*) $\tau = 0.6$ and (*d*) $\tau = 0.0$ (units are arbitrary).

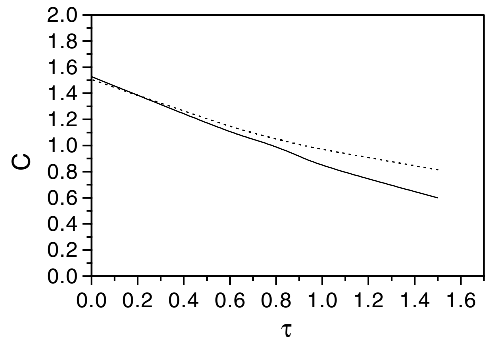

Figure 2. A plot of the speed c of the travelling wavefront solution versus relaxation time τ (analytical, dotted line; numerical, solid line) for $D = 1.0$, $k = 0.6$ (units are arbitrary).

appropriately over a range to generate numerically stable travelling wavefront solution. The interplay of β and k will be considered in more detail in the later part of this section.

In figure 1 we compare the analytical (dotted) and the numerical (solid) solutions corresponding to (22) and (4), respectively, for different values of τ. From our analysis it is apparent that they agree fairly well for τ roughly in the range between 0.1 and 0.5. In figure 1(*d*) we present the result for $\tau = 0$, which corresponds to the typical Fisher case. The analytical curve is marginally steeper than the numerical one. In figure 2 we compare the speed of the travelling wavefront computed numerically from (4) with that obtained analytically following (20) for several values of τ. It follows that they agree reasonably well when $\tau \leqslant k$, i.e. in the range 0.1–0.5. As τ approaches zero the analytical value of c becomes lower than the numerical one. For higher values of τ the disagreement between analytical and numerical values of c grows rapidly.

The above numerical observations suggest that there is a strong interplay of k and τ (or β) in the dynamics so far as the form and stability of the travelling wavefront solution is concerned. To explore this aspect more clearly we now carry out an asymptotic analysis of the problem. To this end we return to equation (7) subject to boundary condition (6). Following Murray we choose the perturbation parameter $\epsilon = 1/c^2$ and look for the asymptotic solution for $0 < \epsilon \ll 1$ by introducing a change of variable $\xi = \frac{z}{c} = \epsilon^{1/2} z$ and $U(z) = g(\xi)$. With these transformations equations (7) and (6), therefore, reduce to

$$\epsilon \frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2} + (n - A + 2Ag)\frac{\mathrm{d}g}{\mathrm{d}\xi} + mnAg(1-g) = 0 \tag{24}$$

and

$$g(-\infty) = 1 \qquad g(+\infty) = 0 \tag{25}$$

respectively. ϵ in the highest derivative in equation (24) identifies it as a singular perturbation problem.

Making use of a regular perturbation series in ϵ

$$g(\xi;\epsilon) = g_0(\xi) + \epsilon g_1(\xi) + \cdots \tag{26}$$

in (24) we obtain after equating the appropriate powers of ϵ

$$(n - A + 2Ag_0)\frac{\mathrm{d}g_0}{\mathrm{d}\xi} = -mnAg_0(1-g_0) \qquad O(1) \tag{27}$$

and

$$(n - A + 2Ag_0)\frac{\mathrm{d}g_1}{\mathrm{d}\xi} + \frac{\mathrm{d}^2 g_0}{\mathrm{d}\xi^2} + 2Ag_1\frac{\mathrm{d}g_0}{\mathrm{d}\xi} + mnAg_1(1-2g_0) = 0 \qquad O(\epsilon). \tag{28}$$

The lowest order equation (27) when integrated yields

$$\ln\left\{\frac{(g_0)^{\beta-k}}{(1-g_0)^{\beta+k}}\right\} = -\beta k\xi + \beta k l \tag{29}$$

where l is a constant of integration. Since we are interested in the solution in the vicinity of $z = 0$, i.e. $\xi = 0$ for which we put $g_0(\xi) = 1/2$, we obtain

$$l = \frac{1}{\beta k}\ln\left\{\frac{\left(\frac{1}{2}\right)^{\beta-k}}{\left(\frac{1}{2}\right)^{\beta+k}}\right\}. \tag{30}$$

Equation (29) precludes the possibility of an explicit solution for $g_0(\xi)$. Depending on β and k we therefore consider three different cases:

(i) $\beta \gg k$ (or $\tau \ll k$)

We have from (30) $l = 0$ and (29) reduces to

$$g_0(\xi) = (1 + \exp(k\xi))^{-1} \qquad \text{or} \qquad U(z) = (1 + \exp(kz/c))^{-1} + O(\epsilon). \tag{31}$$

This is the standard asymptotic solution for $U(z)$ for which the effect of memory is negligible.

(ii) $\beta \approx k$ (i.e. $\tau \approx k$)
We obtain similarly from (29) and (30)

$$g_0(\xi) = \left(1 - \frac{\exp(k\xi/2)}{2}\right) \qquad \text{or} \quad U(z) = \left(1 - \frac{\exp((kz)/(2c))}{2}\right) + O(\epsilon). \tag{32}$$

When both β and k are small compared to 1 and the exponential term in (32) is small, it is easy to put the $O(1)$ term approximately in the form of (31) as

$$U(z) \approx \left(1 + \frac{\exp((kz)/(2c))}{2}\right)^{-1}. \tag{33}$$

(iii) $\beta \ll k$ (i.e. $\tau \gg k$)
We obtain

$$g_0(\xi) = \frac{1 \pm \sqrt{1 - \exp(\beta\xi)}}{2} + O(\epsilon). \tag{34}$$

The form of this solution is generically different from those of (32) and (31) since it is independent of k.

We now employ the above asymptotic solutions to understand the relation between the steepness of the curve and the speed of propagation and the correlation time τ. Since the negative gradient at $z = 0$ using the solutions (31), (32) and (34) gives the steepness (s) of the solutions, we have

$$-U'(0) = s = \frac{k}{4c} \qquad \beta \gg k$$
$$-U'(0) = s = \frac{k}{9c} \qquad \beta \simeq k$$
$$-U'(0) = s = \frac{\beta}{4c} \qquad \beta \ll k.$$

The above relations suggest that steepness goes as $\sim 1/c$, and around $\beta \simeq k$ the steepness is lower than that for the first case ($\beta \gg k$). This makes the analytical solution in this region less steep and brings it closer to the numerical one as well as to the asymptotic solution. For large τ (i.e. $\beta \ll k$) the solution being independent of k tends to differ from the numerical one appreciably.

The three cases discussed above show that monotonic solutions satisfying $U(-\infty) = 1$ and $U(\infty) = 0$ for finite wave speed ($c \geqslant c_{\min}$) exist for the cases (i) and (ii), i.e. when τ is short but finite; $\tau \leqslant k$. The assertion of this asymptotic analysis is in clear agreement with our numerical simulation and our choice of a smaller value of k as discussed earlier.

The aforesaid analysis clearly demonstrates that although the nature of the partial differential equation changes from parabolic to hyperbolic type due to the inclusion of relaxation time, the Fisher equation can be solved by the Murray ansatz [4] to derive the exact wave speed and the travelling wavefront solution for a suitable range of relaxation times τ allowed by the strength of the reaction term. A compromise between the exact and the numerical solutions can be obtained for relatively small reaction terms. The method can be extended further to study other density-dependent diffusive processes.

2.2. The Burgers equation with finite memory transport

The Burgers equation [6] is a simple model of turbulence which illustrates an interaction between convection and diffusion. The convection incorporates nonlinearity in the dynamics. To include the finite memory effect we proceed as follows:

We start with the following functional relation between flux $J(x, t+\tau)$ at a time $t+\tau$ and the field variable $u(x, t)$ and its gradient term at an earlier time t;

$$J(x, t+\tau) = \frac{1}{2}u^2(x, t) - \gamma\frac{\partial u(x, t)}{\partial x} \tag{35}$$

where γ is a constant. Expanding J again up to first order in τ and differentiating the resulting equation with respect to x followed by differentiation of equation (2) for $k = 0$ (i.e. in the absence of any source term) with respect to time t and elimination of J as done in the last section, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \tau\frac{\partial^2 u}{\partial t^2} = \gamma\frac{\partial^2 u}{\partial x^2}. \tag{36}$$

For $\tau = 0$ equation (36) assumes the form of the classical Burgers equation [1, 6] when $u(x, t)$ and γ are identified as the velocity field and kinematic viscosity, respectively.

We now seek a travelling wave solution of the Burgers equation with memory (36) in the form $U(z) = u(x - ct)$, $z = x - ct$, where c is again the wave speed to be determined. This results in the following equation:

$$-\left(\frac{c^2}{\beta} + \gamma\right)\frac{\partial^2 U}{\partial z^2} + U\frac{\partial U}{\partial z} - c\frac{\partial U}{\partial z} = 0 \tag{37}$$

where $\beta = 1/\tau$.

We now impose the bound condition on $U(z)$ that it asymptotically tends to constant values u_1 as $z \to -\infty$ and u_2 as $z \to +\infty$ and $u_1 > u_2$.

A direct integration of (37) yields

$$\frac{\partial U}{\partial z} = \frac{1}{2\left(\frac{c^2}{\beta} + \gamma\right)}(U^2 - 2cU - 2A) \tag{38}$$

where A is the integration constant. If u_1 and u_2 are the roots of the quadratic equation $U^2 - 2cU - 2A = 0$, then the wave speed c and the constant A can be obtained as

$$c = \frac{u_1 + u_2}{2} \qquad \text{and} \qquad A = -\frac{1}{2}u_1u_2. \tag{39}$$

Equation (38) can then be rewritten in the form

$$2\left(\frac{c^2}{\beta} + \gamma\right)\frac{\partial U}{\partial z} = (U - u_1)(U - u_2) \tag{40}$$

to integrate to obtain finally

$$U(z) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2)\tanh\left[\frac{z}{4\delta}\right] \tag{41}$$

where δ is given by

$$\delta = \left(\frac{\frac{c^2}{\beta} + \gamma}{u_1 - u_2}\right). \tag{42}$$

The above analysis shows that the shape of the wave form is not only affected by kinematic viscosity γ but also by an additional contribution c^2/β due to finite relaxation time $\tau(=1/\beta)$ such that $(c^2/\beta) + \gamma$ behaves as the effective kinematic viscosity. It is thus apparent that the balance between the steepening effect of the convection as well as the smoothing effect due to kinematic viscosity is enhanced by the presence of the wave speed dependent term c^2/β. Thus although the wave speed $c[(u_1 + u_2)/2]$ itself remain unaffected by the finite memory effect in contrast to our earlier case of the Fisher equation, the transmission layer thickness 'δ'—which is a measure of shock thickness—increases for higher speed c and relaxation time τ. This implies that as the wave moves faster, the shock smooths out more and more so that the speed dependence of thickness δ makes the dynamics self-regulating in the problem of interaction between convection and diffusion.

3. Conclusions

The existence of relaxation or delay time is an important feature in reaction–diffusion and convection–diffusion systems. In this paper we have shown that two prototypical representatives of these systems, a generalized Fisher equation and the Burgers equation, can be solved exactly for finite arbitrary delay time using conventional methods. While the wave speed is significantly modified in the Fisher problem for finite memory transport, the speed of the travelling wave in the corresponding Burgers problem remains unaffected, delay time being effective in smoothing out the shock-wave nature of the travelling wave. We also establish numerically that for the reaction–diffusion system the strength of the reaction term must not exceed a critical limit to allow travelling wavefront solutions to exist for appreciable memory or relaxation effect. In view of the fact that the studies on reaction–diffusion and convection–diffusion with finite memory transport have been applied to forest fire [21] and population growth models [14], Neolithic transitions [22] and in several other areas under various approximate schemes [10–13, 15–20], we believe that the present exact solutions for the generalized Fisher and Burgers problems are very much pertinent in this context.

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